OSCILLATIONS OF A SEMISPACE WITH A FREE BOUNDARY

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The study of unique features of the earth's seismic structure has entered a new phase of development, due to the change from use of explosions and earthquakes, sources with uncontrolled frequency spectrum and directionality, to use of nonexplosive excitation sources including vibrators. High power surface or subsurface mounted vibrators can produce axially symmetric loading on rock masses (in boreholes, shafts, etc.) which are considered isotropic in the approximation widely used. Thus formal studies of wave processes in simplified models of elastic media become important, in order to obtain a theoretical basis for performing seismic experiments. Discrepancies which appear, the "medium model noise," are due to noncorrespondence between the structure of the actual medium and the model used, and must be considered when interpreting results [1].

From the engineering viewpoint, such techniques are of value in oil and gas prospecting and increasing yields from proven fields, in redistributing stresses in the earth's core (in seismically active regions), and in prevention of mine caveins in heavily mined regions.

Ground level vibration sources produce more intense surface waves than do explosions. However a subsurface controlled source can significantly reduce the level of such waves.

The present study will consider the steady state axially symmetric problem of oscillation of an isotropic semispace produced by an "immersed source," having the form of a circular gap located at some depth parallel to the "daylight" surface, which latter is free of external loading. The problem is of practical importance in connection with the earth vibration sounding program [1]. The same problem for a point source at the point (0, 0, h) was presented in [2]. In the problem to be considered here surface standing waves are absent. The vibration source is a gap with extensive cross section (we assume that vibrations are transferred to the gap from a surface source through a shaft). The problem is of interest to both seismologists using vibration sources, and students of mechanics investigating oscillations of a continuous medium.

1. In cylindrical coordinates (r, φ, z) the problem can be formulated as:

$$\left[\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \frac{\mu}{\lambda+2\mu}\partial_z^2 - \frac{\rho}{\lambda+2\mu}\partial_t^2\right]u_r + \frac{\lambda+\mu}{\lambda+2\mu}\partial_r\partial_z u_z = 0, \tag{1.1}$$

$$\frac{\lambda+\mu}{\mu}\left(\partial_r + \frac{1}{r}\right)\partial_z u_r + \left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{\lambda+2\mu}{\mu}\partial_z^2 - \frac{\rho}{\mu}\partial_t^2\right]u_z = 0;$$

$$(\sigma_z)_{z=0} = 0,\ (\tau_{rz})_{z=0} = 0,\ 0 \leqslant r < \infty,\ 0 \leqslant z < \infty; \tag{1.2}$$

$$(\sigma_z)_{z=h} = -f_k(r)e^{-i\omega t},\ (\tau_{rz})_{z=h} = 0,\quad h > 0\ (k = 1, 2);$$

$$(k = 1)\ r^2 \neq r_0^2 > 0;\quad (k = 2)\ r = r_0;\ 0 \leqslant r < \infty, \tag{1.3}$$

where $u_r(r, z, t)$, $u_z(r, z, t)$ are displacements; the surface $z = 0$ is free of "outcrop" stresses, Eq. (1.2). The planar gap defined by $z = h$, $0 \leqslant r < \infty$ is characterized by the load of Eq. (1.3); ∂_r, ∂_z, ∂_t are partial derivatives with respect to coordinates and time; λ, μ are the Lamé elastic constants; ρ is the density of the medium.

The problem of the semispace oscillations produced by the immersed "gap" source is complicated by the doubling of the boundary conditions, Eqs. (1.2), (1.3). As is well known the complete deformation field consists of "pure" oscillations and traveling waves.

In the present study we will not consider the traveling wave field. The problem of Eqs. (1.1)-(1.3) was solved by use of Hankel integral transforms.

2. A solution in closed form convenient for analytical study was found for the following value of the "boundary" function $f_k(r)$:

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$$f_k(r)=\begin{cases}\dfrac{\omega}{c_2}\dfrac{\mu\beta^{1/2}}{r_0^2-r^2}\left[r_0J_0\left(r\beta^{1/2}\dfrac{\omega}{c_2}\right)J_1\left(r_0\beta^{1/2}\dfrac{\omega}{c_2}\right)-\right.\\ \qquad\left.-rJ_0\left(r_0\beta^{1/2}\dfrac{\omega}{c_2}\right)J_1\left(r\beta^{1/2}\dfrac{\omega}{c_2}\right)\right], \quad k=1,\\ \dfrac{\mu}{2}\beta\dfrac{\omega^2}{c_2^2}\left\{\left[J_0'\left(r_0\beta^{1/2}\dfrac{\omega}{c_2}\right)\right]^2+\left[J_0\left(r_0\beta^{1/2}\dfrac{\omega}{c_2}\right)\right]^2\right\}, \quad r_0=r,\ k=2.\end{cases}$$

Consequently, for $r > r_0$, $f_1(r)$ falls off as $1/r$ and $1/r^{3/2}$ as $r \to \infty$. This permits use of Lommel integrals [3].

The displacements u_r, u_z* in the semispace can be expressed in the form

$$u_r=\frac{\omega}{c_2}e^{-i\omega t}\int_{1/\beta}^{\infty}N_1(z)J_0\left(\frac{\omega}{c_2}\frac{r_0}{\alpha^{1/2}}\right)J_1\left(\frac{\omega}{c_2}\frac{r}{\alpha^{1/2}}\right)\frac{d\alpha}{\alpha^{3/2}},$$
$$u_z=-\frac{\omega}{c_2}e^{-i\omega t}\int_{1/\beta}^{\infty}(\beta\alpha-1)^{1/2}N_2(z)J_0\left(\frac{\omega}{c_2}\frac{r_0}{\alpha^{1/2}}\right)J_0\left(\frac{\omega}{c_2}\frac{r}{\alpha^{1/2}}\right)\frac{d\alpha}{\alpha^{3/2}}, \tag{2.1}$$

where $\beta = c_2^2/c_1^2$; $c_1^2 = (\lambda + 2\mu)/\rho$; $c_2^2 = \mu/\rho$.

The functions $N_1(z)$, $N_2(z)$ have the form

$$N_1(z)=A\left[\frac{2-\alpha}{2}\sin\theta_1 z+(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}\sin\theta_2 z\right]-(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}B\left[\frac{2}{2-\alpha}\cos\theta_1 z-\cos\theta_2 z\right],$$
$$N_2(z)=A\left[\frac{2-\alpha}{2}\cos\theta_1 z-\cos\theta_2 z\right]+B\left[\frac{2}{2-\alpha}(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}\sin\theta_1 z+\sin\theta_2 z\right],$$

where

$$\theta_1=\frac{\omega}{c_2\alpha^{1/2}}(\beta\alpha-1)^{1/2},\quad \theta_2=\frac{\omega}{c_2\alpha^{1/2}}(\alpha-1)^{1/2},$$
$$AM=\frac{4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}}{(2-\alpha)^2}\sin\theta_1 h+\sin\theta_2 h,\quad B=-\frac{\cos\theta_1 h-\cos\theta_2 h}{M}. \tag{2.2}$$

Further $M \equiv M(\alpha, h)$ has the form

$$M=\left[\frac{4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}}{(2-\alpha)^2}\sin\theta_1 h+\sin\theta_2 h\right][(2-\alpha)^2\sin\theta_1 h+$$
$$+4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}\sin\theta_2 h]-4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}(\cos\theta_1 h-\cos\theta_2 h)^2.$$

Equation (2.1) indicates that at $z = 0$ the displacements u_r, u_z are nonzero.

The stresses appearing in the boundary conditions are expressed by

$$\tau_{rz}=\mu\frac{\omega^2}{c_2^2}e^{-i\omega t}\int_{1/\beta}^{\infty}(2-\alpha)(\beta\alpha-1)^{1/2}N_3(z)J_0\left(\frac{\omega}{c_2}\frac{r_0}{\alpha^{1/2}}\right)J_1\left(\frac{\omega}{c_2}\frac{r}{\alpha^{1/2}}\right)\frac{d\alpha}{\alpha^2},$$
$$\sigma_z=-\frac{\mu}{2}\frac{\omega^2}{c_2^2}e^{-i\omega t}\int_{1/\beta}^{\infty}N_4(z)J_0\left(\frac{\omega}{c_2}\frac{r_0}{\alpha^{1/2}}\right)J_0\left(\frac{\omega}{c_2}\frac{r}{\alpha^{1/2}}\right)\frac{d\alpha}{\alpha^2}.$$

The functions $N_3(z)$, $N_4(z)$ are defined by

$$N_3(z)=A(\cos\theta_1 z-\cos\theta_2 z)+B\left[\frac{4}{(2-\alpha)^2}(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}\sin\theta_1 z+\sin\theta_2 z\right],$$
$$N_4(z)=A[(2-\alpha)^2\sin\theta_1 z+4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}\sin\theta_2 z]$$
$$-4(\beta\alpha-1)^{1/2}(\alpha-1)^{1/2}B(\cos\theta_1 z-\cos\theta_2 z).$$

Values of A and B are taken from Eq. (2.2).

The integrands for the displacements of Eq. (2.1) containing a "denominator" $M(\alpha, h)$ with roots α_s having singularities at the point $\alpha = \alpha_s$. Therefore the integrals are to be regarded as main values in the Cauchy sense.

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*Homogeneous solutions of Eqs. (1.1), (1.2) are used.

The solution obtained is applicable to studies of oscillations in rock masses produced by sources located at a specific depth below the free "daylight" surface.

LITERATURE CITED

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